A CORRECTION TO "DYNKIN GAMES VIA DIRICHLET FORMS AND SINGULAR CONTROL OF ONE-DIMENSIONAL DIFFUSION"

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Abstract. In the paper "Dynkin Games Via Dirichlet Forms and Singular Control of One-Dimensional Diffusion"[1], the authors tried to show the existences of a smooth value function and an optimal policy to a one-dimensional stochastic singular control problem, where the underlying process is a generalized diffusion process given by $dX_t = \mu(X_t)dt + \sigma(X_t)dw_t$, in which w_t is a Wiener process. It is found that either a condition on $\mu(x)$ and $\sigma(x)$ should be added, or a different diffusion process should be considered and as a result, the main theorem of this paper should be amended.

Key words. Dynkin game, Dirichlet form, Stochastic singular control

AMS subject classifications. 31C25, 60G40, 60H30, 93E20

In the paper "Dynkin Games Via Dirichlet Forms and Singular Control of One-Dimensional Diffusion"[1], the authors tried to show the existences of a smooth value function and an optimal policy to a one-dimensional stochastic singular control problem via Dynkin game and Dirichlet form. The value function $V(x)$ of a Dynkin game is known to exist [3], which is the solution of a variational inequality problem involving Dirichlet form. The integration of $V(x)$ turns out to be a smooth optimal return function $W(x)$ for a stochastic singular control problem. Thus the traditional technique of viscosity solution is avoided.

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In their paper, the infinitesimal generator is defined as (see page 693, Eq. 4.1 in [1])

$$Lu(x) = \frac{d}{dm}\frac{d}{ds}u(x) = \mu(x)u'(x) + \frac{1}{2}\sigma(x)^2u''(x), \tag{0.1}$$

where $ds(x) = \dot{s}(x)dx, dm(x) = \dot{m}(x)dx$, and (see Eq. 4.2 in [1])

$$\dot{s}(x) = \exp\left(-\int_0^x \frac{2\mu(y)}{\sigma(y)^2}dy\right), \quad \dot{m}(x) = \frac{2}{\sigma(x)^2}\exp\left(\int_0^x \frac{2\mu(y)}{\sigma(y)^2}dy\right). \tag{0.2}$$

The value function $W(x)$ of the stochastic singular control problem is assumed to satisfy the following PDE (see Eq. 3.23 on page 693 in [1])

$$\alpha W(x) - \frac{d}{dm}\frac{d}{ds}W(x) = h(x), \tag{0.3}$$

or equivalently

$$\alpha W(x) - \mu(x)W'(x) - \frac{1}{2}\sigma(x)^2W''(x) = h(x), \tag{0.4}$$

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where (see Eq. 3.21 and Eq. 3.22 in [1])

$$h(x) = \int_0^x H(y)\dot{s}(y)dy + C, \tag{0.5}$$

and

$$W(x) = \int_a^x V(y)\dot{s}(y)dy + \frac{1}{\alpha}\left(-\frac{f_1'(a)}{\dot{m}(a)} + h(a)\right). \tag{0.6}$$

Then in the proof of Theorem 3.2 on page 693 in [1], the authors constructed the function

$$U(x) = \alpha W(x) - \frac{d}{dm}\frac{d}{ds}W(x) - h(x), \tag{0.7}$$

and claimed that

$$\frac{1}{\dot{s}(x)}U'(x) = \alpha V(x) - \frac{d}{ds}\frac{d}{dm}V(x) - H(x). \tag{0.8}$$

This is equivalent to

$$U(x) = \alpha W(x) - \mu(x)W'(x) - \frac{1}{2}\sigma(x)^2 W''(x) - h(x), \tag{0.9}$$

and

$$\frac{1}{\dot{s}(x)}U'(x) = \alpha V(x) - \mu(x)V'(x) - \frac{1}{2}\sigma(x)^2 V''(x) - H(x). \tag{0.10}$$

However, by a careful examination, it can be seen that the above proposition is not true in general. The reason here is that $\mu(x)$ and $\sigma(x)$ are both functions of x, and when taking the derivative of $U(x)$, the product rule has to be applied. The details is shown below.

By the definition of $W(x)$ in (0.6), $h(x)$ in (0.5) and $\dot{s}(x)$ in (0.2), we get

$$\begin{aligned} W'(x) &= V(x)\dot{s}(x), \\ h'(x) &= H(x)\dot{s}(x), \\ \ddot{s}(x) &= -\dot{s}(x)\frac{2\mu(x)}{\sigma(x)^2}, \end{aligned}$$

hence

$$\begin{aligned} W''(x) &= V'(x)\dot{s}(x) - V(x)\dot{s}(x)\frac{2\mu(x)}{\sigma(x)^2}, \\ W'''(x) &= V''(x)\dot{s}(x) - 2V'(x)\dot{s}(x)\frac{2\mu(x)}{\sigma(x)^2} + V(x)\dot{s}\frac{4\mu(x)^2}{\sigma(x)^4} \\ &\quad - V(x)\dot{s}(x)\left(2\mu'(x)\sigma(x)^{-2} - 4\mu(x)\sigma(x)^{-3}\sigma'(x)\right). \end{aligned}$$

Now if we take the derivative of $U(x)$ in (0.9) we get

$$\begin{aligned} U'(x) =& \alpha V(x)\dot{s}(x) - \mu'(x)V(x)\dot{s}(x) - \mu(x)V'(x)\dot{s}(x) + \mu(x)V(x)\dot{s}(x)\frac{2\mu(x)}{\sigma(x)^2} \\ & - \sigma(x)\sigma'(x)V'(x)\dot{s}(x) + \sigma(x)\sigma'(x)V(x)\dot{s}(x)\frac{2\mu(x)}{\sigma(x)^2} \\ & - \frac{1}{2}\sigma(x)^2\Big(V''(x)\dot{s}(x) - 2V'(x)\dot{s}(x)\frac{2\mu(x)}{\sigma(x)^2} + V(x)\dot{s}\frac{4\mu(x)^2}{\sigma(x)^4} \\ & - V(x)\dot{s}(x)\left(2\mu'(x)\sigma(x)^{-2} - 4\mu(x)\sigma(x)^{-3}\sigma'(x)\right)\Big) - H(x)\dot{s}(x). \end{aligned}$$

After simplifying this expression and comparing it to (0.10) we should have the following

$$0 = -\sigma(x)\sigma'(x)V'(x) + 2\mu(x)V'(x), \tag{0.11}$$

which does not hold in general.

A possible correction to this paper is to add a condition

$$2\mu(x) = \sigma(x)\sigma'(x) \tag{0.12}$$

deduced from (0.11), but a second concern of this paper might be more profound.

The Dirichlet form in this paper is defined as (see Eq. 3.3 on page 686 in [1])

$$\mathcal{E}(u,v) = \int_{-A}^{A} u'(x)v'(x)\frac{1}{\dot{m}(x)}dx, \quad u,v \in \mathcal{F}, \tag{0.13}$$

where

$$\begin{aligned}\mathcal{F} &= H^1((-A,A);dx)\\ &= \{u \in L^2((-A,A);dx) : \ u \text{ is absolutely continuous}, u' \in L^2((-A,A);dx)\}.\end{aligned}$$

The authors claimed that this Dirichlet form $(\mathcal{E},\mathcal{F})$ is regular on $L^2([-A,A];ds)$ and the associated underlying process is a reflecting barrier diffusion on $[-A,A]$ with infinitesimal generator $\frac{d}{ds}\frac{d}{dm}$, i.e., the generator L given in (0.1). The correspondence is given by (see Corollary 1.3.1 on page 21 of [2])

$$\mathcal{E}(u,v) = (-Lu, v), \quad u \in \mathcal{D}(L), v \in \mathcal{F}, \tag{0.14}$$

where $\mathcal{D}(L)$ is the domain of L. Since the underlying process is a reflecting barrier diffusion on $[-A,A]$, $\mathcal{D}(L)$ is given by (see page 22 of [2])

$$\begin{aligned}\mathcal{D}(L) = \{u \in \mathcal{F} : u' &\text{ is absolutely continuous},\\ &u'' \in L^2((-A,A);dx), u'(-A) = u'(A) = 0\}.\end{aligned}$$

Now we try the integration by parts on (0.13) and get

$$\begin{aligned}\mathcal{E}(u,v) &= \int_{-A}^{A} u'(x)v'(x)\frac{\sigma(x)^2}{2}\exp\left(-\int_0^x \frac{2\mu(y)}{\sigma(y)^2}dy\right)dx\\ &= -\int_{-A}^{A}\left(\frac{\sigma(x)^2}{2}u''(x) + \sigma(x)\sigma'(x)u'(x) - \mu(x)u'(x)\right)v(x)\exp\left(-\int_0^x \frac{2\mu(y)}{\sigma(y)^2}dy\right)dx.\end{aligned}$$

Once again, when the condition (0.12) holds, we get

$$\frac{\sigma(x)^2}{2}u''(x) + \sigma(x)\sigma'(x)u'(x) - \mu(x)u'(x) = \frac{\sigma(x)^2}{2}u''(x) + \mu(x)u'(x) = Lu(x),$$

and (0.14) holds.

It should also be noticed that there might be a feasible way to amend this issue. We may simply consider the diffusion

$$dX_t = \gamma(X_t)dt + \sigma(X_t)dw_t,$$

with the infinitesimal generator

$$L_\gamma u(x) = \gamma(x)u'(x) + \frac{1}{2}\sigma(x)^2 u''(x),$$

and put

$$\gamma(x) = \sigma(x)\sigma'(x) - \mu(x).$$

The Dirichlet form is still defined as in (0.13) and $\dot{s}(x), \dot{m}(x)$ are still given in (0.2), then we get

$$\mathcal{E}(u,v) = (-L_\gamma u, v).$$

It might be interesting to notice that when σ is a constant, we get $\gamma(x) = -\mu(x)$, and

$$\mathcal{E}(u,v) = \int_{-A}^{A} u'(x)v'(x)\frac{\sigma^2}{2}\exp\left(\int_0^x \frac{2\gamma(y)}{\sigma^2}dy\right)dx. \tag{0.15}$$

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